# On the uniqueness of continuous inverse kinetic theory for incompressible fluids

Massimo Tessarotto[1,2] and Marco Ellero[3]

[1]*Department of Mathematics and Informatics,*

*University of Trieste, Trieste, Italy*

[2] *Consortium for Magnetofluid Dynamics, Trieste, Italy*

[4] *Lehstuehl fuer Aerodynamik, Technische Universitaet Munchen, Munchen, Germany*

## Abstract

Fundamental aspects of inverse kinetic theories for incompressible Navier-Stokes equations concern the possibility of defining uniquely the kinetic equation underlying such models and furthermore, the construction of a kinetic theory implying also the energy equation. The latter condition is consistent with the requirement that fluid fields result classical solutions of the fluid equations. These issues appear of potential relevance both from the mathematical viewpoint and for the physical interpretation of the theory. In this paper we intend to prove that the non-uniqueness feature can be resolved by imposing suitable assumptions. These include, in particular, the requirement that the kinetic equation be equivalent, in a suitable sense, to a Fokker-Planck kinetic equation. Its Fokker-Planck coefficients are proven to be uniquely determined by means of appropriate prescriptions. In addition, as a further result, it is proven that the inverse kinetic equation satisfies both an entropy principle and the energy equation for the fluid fields.

## I.  INTRODUCTION

An aspect of fluid dynamics is represented by the class of so-called *inverse problems*, involving the search of model kinetic theories able to yield a prescribed set of fluid equations, with particular reference to the continuity and Navier-Stokes (N-S) equations for Newtonian fluids, by means of suitable velocity-moments of an appropriate kinetic distribution function $f(\mathbf{r}, \mathbf{v}, t)$. Among such model theories, special relevance pertains to those describing, self-consistently, isothermal incompressible fluids described by the so-called incompressible Navier-Stokes equations (INSE) in the sense that the fluid equations are satisfied for arbitrary fluid fields and for arbitrary initial conditions for the kinetic distribution. In particular the fluid fields are in this case identified with $\{\rho \equiv \rho_o, \mathbf{V}, p\}$, being $\mathbf{V}$ the fluid velocity and, $\rho$ and $p$ respectively, the mass density and fluid pressure, both non-negative in the closure $(\overline{\Omega})$ of the fluid domain $\Omega \subseteq \mathbb{R}^3$, while $\rho$ is always constant (condition of incompressibility). A desirable feature of the theory is, however, the requirement that the relevant inverse kinetic equation are uniquely defined *(in particular, by imposing that a particular solution is provided by local Maxwellian equilibria)* as well as the possibility of requiring, besides INSE, also additional fluid equations, to be satisfied by means of suitable moment equations (*extended INSE*). An example is provided by the energy equation, i.e., the fluid equation obtained by taking the scalar product of the Navier-Stokes equation by the fluid velocity $\mathbf{V}$

$$\frac{\partial}{\partial t}\frac{V^2}{2} + \mathbf{V} \cdot \nabla \frac{V^2}{2} + \frac{1}{\rho_o}\mathbf{V} \cdot \nabla p + \frac{1}{\rho_o}\mathbf{V} \cdot \mathbf{f} - \nu \mathbf{V}\cdot\nabla^2\mathbf{V} = \mathbf{0}, \tag{1}$$

where $\mathbf{f}$ is the volume force density acting on the fluid element. In fact, it is well known that the energy equation is not satisfied by weak solutions of INSE and, as a consequence, also by certain numerical solutions, such as those based on weak solutions such as possibly so-called finite-volume schemes. Therefore, imposing its validity for the inverse kinetic equation yields is a necessary condition for the validity of classical solutions for INSE. In a previous work [1, 2, 3], an explicit solution to INSE has been discovered based on a continuous inverse kinetic theory, adopting a "Vlasov" differential kinetic equation defined by a suitable streaming operator $L$. Basic feature of this kinetic equation is that, besides yielding INSE as moment equations, it allows, as particular solutions, local kinetic equilibria for arbitrary (but suitably smooth) fluid fluids $\{\rho_o, \mathbf{V}, p\}$. However, as pointed out in Refs.[3, 4], the inverse kinetic equation defined in this way results parameter-dependent and hence non-unique, even in the case of local Maxwellian kinetic equilibria. This non-uniqueness feature

may result as a potentially undesirable feature of the mathematical model, since it prevents the possible physical interpretation of the theory (in particular, of the mean-field force **F**) and may result inconvenient from the mathematical viewpoint since the free parameter may be chosen, for example, arbitrarily large in magnitude. It is therefore highly desirable to to eliminate it from the theory [4].

The purpose of this paper is to present a reformulation of the problem which permits to cast the inverse kinetic equation in a form which results unique, thus eliminating possible parameter-dependences in the relevant streaming operator. Actually, the prescription of uniqueness on the kinetic equation is to be intended in the a suitably meaningful, i.e., to hold under the requirement that the relevant set of fluid equations are fulfilled identically by the fluid fields in the extended domain $\Omega \times I$. This means that arbitrary contributions in the kinetic equation, which vanish identically under a such an hypothesis, can be included in the same kinetic equation. Consistent with the previous regularity assumption, here we intend to consider, in particular, the requirement that the inverse kinetic equation yields also the energy equation (1). Note that this new formulation of the inverse kinetic theory is also relevant for comparisons both with previous literature dealing with the determination of the probability distribution function (PDF) for incompressible fluids [5] and with emerging theoretical approaches for the determination of the PDF for small scale turbulence to model homogeneous and isotropic turbulence in the inertial range. As further development of the theory, it is shown that the streaming operator can be suitably and uniquely modified in such a way that the inverse kinetic equation yields the extended INSE equations, i.e., besides the incompressible Navier-Stokes equations also the energy equation. In particular we intend to prove that the mean-field force **F** can be uniquely defined in such a way that both kinetic equilibrium and moment equations yield uniquely such equations.

## II.   NON-UNIQUENESS OF THE STREAMING OPERATOR

We start recalling that the inverse kinetic equation which is assumed of the form

$$L(\mathbf{F})f = 0 \tag{2}$$

[3]. In particular, the streaming operator $L$ is assumed to be realized by a differential operator of the form $L(\mathbf{F}) = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{v}} \cdot \{\mathbf{F}\}$. The vector field **F** (*mean-field force*) can

be assumed of the form $\mathbf{F} \equiv \mathbf{F}_0 + \mathbf{F}_1$, where $\mathbf{F}_0$ and $\mathbf{F}_1$, requiring in particular that they depend on the minimal number of velocity moments [4], can be defined as follows

$$\mathbf{F}_0(\mathbf{x},t;f) = \frac{1}{\rho_o}\left[\nabla\cdot\underline{\underline{\mathbf{\Pi}}} - \nabla p_1 - \mathbf{f}\right] + \mathbf{u}\cdot\nabla\mathbf{V} + \nu\nabla^2\mathbf{V}, \tag{3}$$

$$\mathbf{F}_1(\mathbf{x},t;f) = \frac{1}{2}\mathbf{u}\left\{\frac{D}{Dt}\ln p_1 + \frac{1}{p_1}\nabla\cdot\mathbf{Q} - \frac{1}{p_1^2}\nabla p\cdot\mathbf{Q}\right\} + \frac{v_{th}^2}{2p_1}\nabla p\left\{\frac{u^2}{v_{th}^2} - \frac{3}{2}\right\}. \tag{4}$$

Here $\mathbf{Q}$ and $\underline{\underline{\mathbf{\Pi}}}$ are respectively the relative kinetic energy flux and the pressure tensor $\mathbf{Q} = \int d^3v\,\mathbf{u}\frac{u^2}{3}f$, $\underline{\underline{\mathbf{\Pi}}} = \int d^3v\,\mathbf{u}\mathbf{u}f$. As a consequence, both $\mathbf{F}_0$ and $\mathbf{F}_1$ are functionally dependent on the kinetic distribution function $f(\mathbf{x},t)$. Supplemented with suitable initial and boundary conditions and subject to suitable smoothness assumptions for the kinetic distribution function $f(\mathbf{x},t)$, several important consequences follow [3]:

- the fluid fields $\{\rho_o, \mathbf{V}, p\}$ can be identified in the whole fluid domain $\Omega$ with suitable velocity moments (which are assumed to exist) of the kinetic distribution function $f(\mathbf{x},t)$ [or equivalent $\widehat{f}(\mathbf{x},t)$], of the form $M_G(r,t) = \int d^3v\,G(\mathbf{x},t)f(\mathbf{x},t)$, where $G(\mathbf{x},t) = 1, \mathbf{v}, E \equiv \frac{1}{3}u^2, \mathbf{v}E, \mathbf{u}\mathbf{u}$, and $\mathbf{u} \equiv \mathbf{v} - \mathbf{V}(\mathbf{r},t)$ is the relative velocity. Thus, we require respectively $\rho_o = \int d^3v\,f(\mathbf{x},t)$, $\mathbf{V}(\mathbf{r},t) = \frac{1}{\rho}\int d^3v\,\mathbf{v}f(\mathbf{x},t)$, $p(\mathbf{r},t) = p_1(\mathbf{r},t) - P_o$ $p_1(\mathbf{r},t)$ being the scalar kinetic pressure, i.e., $p_1(\mathbf{r},t) = \int d\mathbf{v}\frac{u^2}{3}f(\mathbf{x},t)$. Requiring, $\nabla p(\mathbf{r},t) = \nabla p_1(\mathbf{r},t)$ and $p_1(\mathbf{r},t)$ strictly positive, it follows that $P_o$ is an arbitrary strictly positive function of time, to be defined so that the physical realizability condition $p(\mathbf{r},t) \geq 0$ is satisfied everywhere in $\overline{\Omega} \times I$ ($I \subseteq \mathbb{R}$ being generally a finite time interval);

- $\{\rho_o, \mathbf{V}, p\}$ are advanced in time by means of the inverse kinetic equation Eq.(2);

- By appropriate choice of the mean-field force $\mathbf{F}$, the moment equations can be proven to satisfy identically INSE, and in particular the Poisson equation for the fluid pressure, as well the appropriate initial and boundary conditions (see Ref.[3]);

- The mean-field force $\mathbf{F}$ results by construction function only of the velocity moments $\{\rho_o, \mathbf{V}, p_1, \mathbf{Q}, \underline{\underline{\mathbf{\Pi}}}\}$, to be denoted as *extended fluid fields*.

- In particular, $L(\mathbf{F})$ can be defined in such a way to allow that the inverse kinetic equation (2) admits, as a particular solution, the local Maxwellian distribution

$$f_M(\mathbf{x},t;\mathbf{V},p_1) = \frac{\rho_0^{5/2}}{(2\pi)^{\frac{3}{2}} p_1^{\frac{3}{2}}} \exp\left\{-X^2\right\}.$$ Here, the notation is standard [3], thus $X^2 = \frac{u^2}{v_{th}^2}$, $v_{th}^2 = 2p_1/\rho_o, p_1$ being the kinetic pressure.

Let us now prove that the inverse kinetic equation defined above (2) is non-unique, even in the particular case of local Maxwellian kinetic equilibria, due to the non-uniqueness in the definition of the mean-field force $\mathbf{F}$ and the streaming operator $L(\mathbf{F})$. In fact, let us introduce the parameter-dependent vector field $\mathbf{F}(\alpha)$

$$\mathbf{F}(\alpha) = \mathbf{F} + \alpha\mathbf{u}\cdot\nabla\mathbf{V} - \alpha\nabla\mathbf{V}\cdot\mathbf{u} \equiv \mathbf{F}_0(\alpha) + \mathbf{F}_1 \tag{5}$$

where $\mathbf{F} \equiv \mathbf{F}(\alpha = 0)$, $\alpha \in \mathbb{R}$ is arbitrary and we have denoted

$$\mathbf{F}_0(\alpha) = \mathbf{F}_0 - \alpha\Delta\mathbf{F}_0 \equiv \mathbf{F}_{0a} + \Delta_1\mathbf{F}_0(\alpha),$$
$$\Delta\mathbf{F}_0 \equiv \mathbf{u}\cdot\nabla\mathbf{V} - \nabla\mathbf{V}\cdot\mathbf{u}, \tag{6}$$
$$\Delta_1\mathbf{F}_0(\alpha) \equiv (1+\alpha)\mathbf{u}\cdot\nabla\mathbf{V} - \alpha\nabla\mathbf{V}\cdot\mathbf{u},$$

where $\mathbf{F}_0$ and $\mathbf{F}_1$ given by Eqs.(3),(4). Furthermore, here we have introduced also the quantity $\Delta_1\mathbf{F}_0(\alpha)$ to denote the parameter-dependent part of $\mathbf{F}_0(\alpha)$. In fact, it is immediate to prove the following elementary results:

a) for arbitrary $\alpha \in \mathbb{R}$, the local Maxwellian distribution $f_M$ is a particular solution of the inverse kinetic equation (2) if and only if the incompressible N-S equations are satisfied;

b) for arbitrary $\alpha$ in $\mathbb{R}$, the moment equations stemming from the kinetic equation (2) coincide with the incompressible N-S equations;

c) the parameter $\alpha$ results manifestly functionally independent of the kinetic distribution function $f(\mathbf{x}, t)$.

The obvious consequence is that the functional form of the vector field $\mathbf{F}_0$, and consequently $\mathbf{F}$, which characterizes the inverse kinetic equation (2) is not unique. The non-uniqueness in the contribution $\mathbf{F}_0(\alpha)$ is carried by the term $\alpha\Delta\mathbf{F}_0$ which does not vanish even if the fluid fields are required to satisfy identically INSE in the set $\Omega \times I$. We intend to show in the sequel that the value of the parameter $\alpha$ can actually be uniquely defined by a suitable prescription on the streaming operator and the related mean-field force.

### III. A UNIQUE REPRESENTATION

To resolve the non-uniqueness feature of the functional form of the streaming operator $L$, due to this parameter dependence, let us now consider again the inverse kinetic equation (2). We intend to prove that the mean-field force $\mathbf{F}$, and in particular the vector field $\mathbf{F}_0(\alpha)$, can be given an unique representation in terms of a suitable set of fluid fields $\{\rho_o, \mathbf{V}, p_1, \mathbf{Q}, \underline{\mathbf{\Pi}}\}$ defined above by introducing a symmetrization condition on the mean field force $\mathbf{F}_0(\alpha)$. To reach this conclusion it is actually sufficient to impose that the kinetic energy flux equation results parameter-independent and suitably defined. Thus, let us consider the moment equation which corresponds the kinetic energy flux density $G(\mathbf{x}, t) = \mathbf{v}\frac{u^2}{3}$. Requiring that $f(\mathbf{x},t)$ is an arbitrary particular solution of the inverse kinetic equation (not necessarily Maxwellian) for which the corresponding moment $\mathbf{q} = \int d^3v \mathbf{v}\frac{u^2}{3}f$ (kinetic energy flux vector) does not vanish identically, the related moment equation takes the form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int d\mathbf{v} G(\mathbf{x},t)f + \nabla \cdot \int d\mathbf{v}\mathbf{v}G(\mathbf{x},t)f- \\
&- \int d\mathbf{v}\left[\mathbf{F}_{0a} + \Delta_1\mathbf{F}_0(\alpha) + \mathbf{F}_1\right]\cdot\frac{\partial G(\mathbf{x},t)}{\partial \mathbf{v}}f- \\
&- \int d\mathbf{v}f\left[\frac{\partial}{\partial t}G(\mathbf{x},t) + \mathbf{v}\cdot\nabla G(\mathbf{x},t)\right] = 0.
\end{aligned}
\tag{7}
$$

Introducing the velocity moments $p_2 = \int d\mathbf{v}\frac{u^4}{3}f$, $\underline{\mathbf{P}} = \int d\mathbf{v}\mathbf{u}\mathbf{u}\frac{u^2}{3}f$ and $\underline{\underline{\mathbf{T}}} = \int d\mathbf{v}\mathbf{u}\mathbf{u}\mathbf{u}f$, the kinetic energy flux equation contains contributions which depend linearly on the undetermined parameter $\alpha$. The contribution to the rate-of-change of $\mathbf{q}$ produced by $\Delta_1\mathbf{F}_0(\alpha)$, which results proportional both to the velocity gradient $\nabla\mathbf{V}$ and the relative kinetic energy flux $\mathbf{Q}$, reads

$$
\mathbf{M}_\alpha(f) \equiv -(1+\alpha)\mathbf{Q}\cdot\nabla\mathbf{V} + \alpha\nabla\mathbf{V}\cdot\mathbf{Q}.
\tag{8}
$$

In order to eliminate the indeterminacy of $\alpha$, since $\alpha$ cannot depend on the kinetic distribution function $f$, a possible choice is provided by the assumption that $\mathbf{M}_\alpha(f)$ takes the symmetrized form

$$
\mathbf{M}_\alpha(f) = -\frac{1}{2}\nabla\mathbf{V}\cdot\mathbf{Q} + \frac{1}{2}\mathbf{Q}\cdot\nabla\mathbf{V},
\tag{9}
$$

which manifestly implies $\alpha = 1/2$. Notice that the symmetrization condition can also be viewed as a constitutive equation for the rate-of-change of the kinetic energy flux vector. In this sense, it is analogous to similar symmetrized constitutive equations adopted in customary approaches to extended thermodynamics [12]. On the other hand, Eq.(9) implies

$\mathbf{M}_\alpha(f) = \frac{1}{2}\mathbf{Q}\times\xi$, $\xi = \nabla \times \mathbf{V}$ being the vorticity field. Thus, $\mathbf{M}_\alpha(f)$ can also be interpreted as the rate-of-change of the kinetic energy flux vector $\mathbf{Q}$ produced by vorticity field $\xi$. From Eq.(9) it follows that $\mathbf{F}_0(\alpha)$ reads

$$\mathbf{F}_0(\alpha = \frac{1}{2}) = \frac{1}{\rho_o} \left[ \nabla{\cdot}\underline{\underline{\mathbf{\Pi}}} - \nabla p_1 - \mathbf{f} \right] + \frac{1}{2}\left( \mathbf{u} \cdot \nabla\mathbf{V}+\nabla\mathbf{V} \cdot \mathbf{u} \right) + \nu\nabla^2\mathbf{V}. \qquad (10)$$

Hence, the functional form of the streaming operator results uniquely determined. As a result of the previous considerations, it is possible to establish the following uniqueness theorem:

**THEOREM 1 – Uniqueness of the Vlasov streaming operator $L(\mathbf{F})$**

*Let us assume that:*

*1) the fluid fields $\{\rho, \mathbf{V}, p\}$ and volume force density $\mathbf{f}(\mathbf{r}, \mathbf{V}, t)$ belong respectively to the functional settings $\{\mathbf{V}(\mathbf{r},t), p(\mathbf{r},t) \in C^{(0)}(\overline{\Omega} \times I), \mathbf{V}(\mathbf{r},t), p(\mathbf{r},t) \in C^{(2,1)}(\Omega \times I)\}$ and $\{\mathbf{f}(\mathbf{r}, \mathbf{v}, t) \in C^{(0)}(\overline{\Omega} \times I), \mathbf{f}(\mathbf{r},t) \in C^{(1,0)}(\Omega \times I)\}$;*

*2) the operator $L(\mathbf{F})$, defining the inverse kinetic equation (2), has the form of the Vlasov streaming operator $L$;*

*3) the solution, $f(\mathbf{x}, t)$, of the inverse kinetic equation (2) exists, results suitably smooth in $\Gamma \times I$ and its velocity moments $\{\rho_o, \mathbf{V}, p_1, \mathbf{Q}, \underline{\underline{\mathbf{\Pi}}}\}$ define the fluid fields $\{\rho_o, \mathbf{V}, p\}$ which are classical solutions of INSE, together with Dirichlet boundary conditions and initial conditions. In addition, the inverse kinetic equation admits, as particular solution, the local Maxwellian distribution $f_M$;*

*4) the mean-field force $\mathbf{F}(\alpha)$ is a function only of the extended fluid fields $\{\rho_o, \mathbf{V}, p_1, \mathbf{Q}, \underline{\underline{\mathbf{\Pi}}}\}$, while the parameter $\alpha$ does not depend functionally on $f(\mathbf{x}, t)$;*

*5) the vector field $\Delta_1 F_0(\alpha)$ satisfies the the symmetry condition (9).*

*Then it follows that the mean-field force $\mathbf{F}$ in the inverse kinetic equation (2) is uniquely defined in terms of $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1$, where the vector fields $\mathbf{F}_0$ and $\mathbf{F}_1$ are given by Eqs. (10) and (4).*

*PROOF*

Let us consider first the case in which the distribution function $f(\mathbf{x}, t)$ coincides with the local Maxwellian distribution $f_M$. In this case by definition the moments $\mathbf{Q}, \underline{\underline{\mathbf{\Pi}}}$ vanish identically while, by construction the mean mean-field force is given by $\mathbf{F}(\alpha)$ [see Eq.(5)], $\alpha \in \mathbb{R}$ being an arbitrary parameter. Let us now assume that $f(\mathbf{x}, t)$ is non-Maxwellian and

that its moment $\mathbf{M}_\alpha(f)$ defined by Eq.(8) is non-vanishing. In this case the uniqueness of $\mathbf{F}$ follows from assumptions 4 and 5. In particular the parameter $\alpha$ is uniquely determined by the symmetry condition (9) in the moment $\mathbf{M}_\alpha(f)$. Since by assumption $\alpha$ is independent of $f(\mathbf{x},t)$ the result applies to arbitrary distribution functions (including the Maxwellian case). Let us now introduce the vector field $\mathbf{F}'= \mathbf{F}+ \Delta\mathbf{F}$, where the vector field $\Delta\mathbf{F}$ is assumed to depend functionally on $f(\mathbf{x},t)$ and defined in such a way that:

A) the kinetic equation $L(\mathbf{F}')f(\mathbf{x},t) = 0$ yields an inverse kinetic theory for INSE, satisfying hypotheses 1-5 of the present theorem, and in particular it produces the same moment equation of the inverse kinetic equation (2) for $G(\mathbf{x},t) = 1, \mathbf{v},E \equiv \frac{1}{3}u^2$;

B) there results identically $\Delta\mathbf{F}(f_M) \equiv 0$, i.e., $\Delta\mathbf{F}$ vanishes identically in the case of a local Maxwellian distribution $f_M$.

Let us prove that necessarily $\Delta\mathbf{F}(f) \equiv 0$ also for arbitrary non-Maxwellian distributions $f$ which are solutions of the inverse kinetic equation. First we notice that from A and B, due to hypotheses 3 and 4, it follows that $\Delta\mathbf{F}$ must depend linearly on $\mathbf{Q}, \underline{\underline{\mathbf{\Pi}}} - p_1\underline{\mathbf{1}}$. On the other hand, again due to assumption A the vector field $\Delta\mathbf{F}$ must give a vanishing contribution to the moments the kinetic equation evaluated with respect to $G(\mathbf{x},t) = 1, \mathbf{v},E \equiv \frac{1}{3}u^2$. Hence, in order that also $\mathbf{F}'$ depends only on the moments $\{\rho_o, \mathbf{V},p_1, \mathbf{Q}, \underline{\underline{\mathbf{\Pi}}}\}$ (hypothesis 4) necessarily it must result $\Delta\mathbf{F}(f) \equiv 0$ also for arbitrary non-Maxwellian distributions $f$.

## IV.  FULFILLMENT OF THE ENERGY EQUATION

As a further development, let us now impose the additional requirement that the inverse kinetic theory yields explicitly also the energy equation (1). We intend to show that the kinetic equation fulfilling such a condition can be obtained by a unique modification of the mean-field force $\mathbf{F} \equiv \mathbf{F}_0(\mathbf{x},t) + \mathbf{F}_1(\mathbf{x},t)$, in particular introducing a suitable new definition of the vector field $\mathbf{F}_1(\mathbf{x},t)$. The appropriate new representation is found to be

$$\mathbf{F}_1(\mathbf{x},t; f) = \frac{1}{2}\mathbf{u}\frac{\partial \ln p_1}{\partial t} - \frac{1}{p_1}\mathbf{V}\cdot\left\{\frac{\partial}{\partial t}\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{V} + \frac{1}{\rho_o}\mathbf{f} - \nu\nabla^2\mathbf{V} + \right.$$

$$\left. +\frac{1}{p_1}\nabla\cdot\mathbf{Q} - \frac{1}{p_1^2}\nabla p\cdot\mathbf{Q}\right\} + \frac{v_{th}^2}{2p_1}\nabla p\left\{\frac{u^2}{v_{th}^2} - \frac{3}{2}\right\} \tag{11}$$

As a consequence, the following result holds:

**THEOREM 2 – Inverse kinetic theory for extended INSE**

*Let us require that:*

*1) assumptions 1-3 of Thm.1 are valid;*

*2) the mean-field* **F** *is defined in terms of* **F**$_0$ *and* **F**$_1$ *given by Eqs. (3) and (11).*

*Then it follows that:*

*A)* $\{\rho, \mathbf{V}, p\}$ *are classical solutions of extended INSE in* $\Omega \times I$ *if and only if the Maxwellian distribution function* $f_M$ *is a particular solution of the inverse kinetic equation (2);*

*B) provided that the solution* $f(\mathbf{x},t)$ *of the inverse kinetic equation (2) exists in* $\Gamma \times I$ *and results suitably summable in the velocity space U, so that the moment equations of (2) corresponding to the weight-functions* $G(\mathbf{x},t) = 1, \mathbf{v}, E \equiv \frac{1}{3}u^2$ *exist, they coincide necessarily with extended INSE.*

*C) the two representations (4) and (11) for* **F**$_1$ *coincide identically*

*PROOF:*

A) The proof is straightforward. In fact, recalling Thm.1, in [3], we notice that Eqs. (11) and (4) manifestly coincide if and only if the energy equation (1) is satisfied identically, i.e., if the fluid fields are solutions of extended INSE.

B) The first two moment equations corresponding to $G(\mathbf{x},t) = 1, \mathbf{v}$ are manifestly independent of the form of **F**$_1$, both in the case of Maxwellian and non-Maxwellian distributions, i.e., (11) and (4). Hence, in such a case Thm.3 of [3] applies, i.e., the moment equations yield INSE. Let us consider, in particular, the third moment equation corresponding to $G(\mathbf{x},t) = \frac{1}{3}u^2$,

$$\frac{\partial}{\partial t}p_1 + \nabla \cdot \mathbf{Q} + \nabla \cdot [\mathbf{V}p_1] - \frac{2}{3}\int d\mathbf{v}\mathbf{F}(\mathbf{x},t)\mathbf{u}f + \frac{2}{3}\nabla\mathbf{V} : \underline{\underline{\mathbf{\Pi}}} = 0. \tag{12}$$

Invoking Eqs. (10) and (11) for **F**$_0$ and **F**$_1$, the previous equation reduces to $p_1\nabla \cdot \mathbf{V} = 0$ if and only if the energy equation (1) is satisfied. Since by construction $p_1 > 0$, this yields the isochoricity condition $\nabla \cdot \mathbf{V} = 0$.

C) Finally, since thanks to A) $\{\rho, \mathbf{V}, p\}$ are necessarily classical solutions of INSE, it follows that they fulfill necessarily also the energy equation (1). Hence, (4) and (11) coincide identically in $\Gamma \times I$.

We conclude that (10) and (11) provide a new form of the inverse kinetic equation applying also to non-Maxwellian equilibria, which results alternative to that given earlier in [3]. The new form applies necessarily to classical solutions. Since weak solutions (and hence

9

possibly also numerical solutions) of INSE may not satisfy exactly the energy equation, the present inverse kinetic theory based on the new definition given above [see Eq.(4)] for the vector field $\mathbf{F}(\mathbf{x},t)$ provides a necessary condition for the existence of strong solutions of INSE. The result seems potentially relevant both from the conceptual viewpoint in mathematical research and for numerical applications.

## V.   CONCLUSIONS

In this paper the non-uniqueness of the definition of the inverse kinetic equation defined by Ellero and Tessarotto (see [3]) has been investigated, proving that the mean-field force $\mathbf{F}$ characterizing such an equation depends on an arbitrary real parameter $\alpha$. To resolve the indeterminacy, a suitably symmetrization condition has been introduced for the kinetic energy flux moment equation. As a consequence, the functional form the mean-field force $\mathbf{F}$ which characterizes the inverse kinetic equation results uniquely determined.   Furthermore, we have proven the positivity of the kinetic distribution function. An open issue remains, however, whether the inverse kinetic equation (2) satisfies an H-theorem, i.e., the entropy results a mononically increasing function of time. Finally, as an additional development, we have shown that, consistently with the assumption that the fluid fields are strong solutions of INSE, the mean-field force can be expressed in such a way to satisfy explicitly also the energy equation.   The result appears significant from the mathematical viewpoint, the physical interpretation of the theory and potential applications to the investigation of complex fluids, such as for example those treated in [6, 10, 11]). In fact, it proves that the inverse kinetic theory developed in [3] can be given an unique form which applies to classical solutions of INSE.

Consortium (Consorzio di Magnetofluidodinamica, Trieste, Italy).

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